The Nucleus 56, No. 4 (2019) 131-136

www.thenucleuspak.org.pk

The Nucleus

ISSN 0029-5698 (Print)

ISSN 2306-6539 (Online)

Characteristics of Dispersed Latin Squares

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ARTICLE INFO

Article history:

Received: *12 September, 2019*

Accepted: *11 April, 2020*

Published: *12 June, 2020*

Keywords:

DoE,

Latin square,

Diamond dispersed pattern,

Knut-Vik square,

Randomization

ABSTRACT

Experimental designs widely use Latin squares for controlled testing to understand causal processes. Among many existing Latin squares for a given order, a design of experiment may need a specific Latin square that offers its treatments (symbols) scattered at the desired locations. There is a need to study systematic dispersion of symbols in Latin squares to assist effective research designs. This paper describes the attributes and applications of one of such designs known as Latin square of diamond dispersed pattern

1. Introduction

In a Design of Experiments (DoE) study, a controlled experiment may manipulate testing of a single variable at a time. The Latin square is a type of experimental design which is formed as a square array of n different elements with each symbol occurring exactly once in each row and each column. It is an unbalanced and fractional factorial experimental design. It helps to control two (or more) blocking factors.

Fractional or partial factorial designs are good alternatives to the full-factorial designs that are time, energy and cost prohibitive [1]. They run faster for a chosen subset of factors level combinations, but this may leave many effects and interactions confounded [2]. The best-known systematic Latin squares (LS) for experimental designs are Knut Vik squares [3] that unfortunately do not exist for Latin squares of even orders [4].

A Knut Vik square (KV square) is a type of design that forms a Latin square of order n with an additional attribute of having each symbol once in both left and right diagonals. These designs help to eliminate source of variation in four directions.

Diamond Dispersion Pattern (DDP) [5] in Latin squares is committed to keep similar symbols distant from each other. The distance between any two similar symbols is maintained throughout the Latin square systematically by using a defined formula. This distance (Manhattan distance) is referred as the dispersion size of a Latin square of diamond-dispersed pattern. According to the calculation of DDP algorithm, for all the even dispersion sizes the minimum order required for a Latin square is an odd number and the resultant Latin square is a KV square. Conversely, DDP of all odd dispersion sizes require even number of symbols as a minimum order of the Latin square. This kind of LS design exploits the feature of KV square for some orders and it also offers systematic arrangement for a Latin square of other orders.

Fisher [6] opposed the exclusive use of a systematic design like KV square and preferred the use of other randomly chosen designs. Francis and Mercia [7] mentioned two major observations regarding KV squares. First, KV square is not suggested in experimental designs that aim to balance out the spatial correlation. The main reason behind this argument is that the repetition of a symbol always carries same common neighboring symbols around it. This kind of plan may fail to overcome variation factors that emerge by spatial interactions. Secondly, randomization is always preferred in experimental design but a KV square being a systematic design does not allow many choices to adjust symbols, rows or columns in a Latin square. Limited randomization restricts control of variation.

Commonly, not all the experimental designs suffer confounding because of spatial correlation, or the spatial interaction may be quite ignorable. Randomization is normally applied in selection of a design among all available designs. The randomization may be applied in assignment of levels of the row/column factor to the rows/columns or in assignment of treatments to the treatment letters. In some environments, randomizations may not help to change the results of an experiment [8].

This paper presents physical layout, attributes and applicability of DDP. It is shown that DDP is a special design of experiment that is useful in situations where same treatments cannot follow each other very soon. In such environment time is the major consideration for implied blocking factors. By using DDP a qualified yield is expected from the experiments in which repetition of treatments need a natural and periodic dispersion. Moreover, DDP offers a

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proper randomization space for the selection of experimental design among sufficiently available number of designs.

2. Geometry of the Pattern

Generating customized Latin squares by using different methods is very common [9]. Our customization of Latin squares is named DDP (Diamond Dispersed Pattern). The name of this pattern refers to its physical layout in a Latin square.

A grid of 7×7 cells having a central cell represented by O is shown in Fig. 1(a). There are only four cells (shaded) around O that are at a Manhattan distance (MD) of one cell from it. This forms a small diamond like shape with radius 1 (MD from the center). The area of this diamond covers total five cells (four boundary cells and one central cell). There are added eight new cells (shaded) at a Manhattan distance of two from the center (Fig. 1(b)). It forms a bigger diamond whose area include new shaded cells and all the cells covered by its internal diamond of radius one. Total number of cells covered by this diamond of radius two are equal to thirteen (4×2 + 4×1 + 1).

Similarly, there will be twelve (4×3) new cells that are exactly at MD of three from the center (Fig. 1(c)) and the new diamond of radius three will covers all newly added cells (boundary cells) and cells of its internal diamonds. There are total twenty-five (4×3 + 4×2 + 4×1 +1) cells covered by this diamond of radius three.

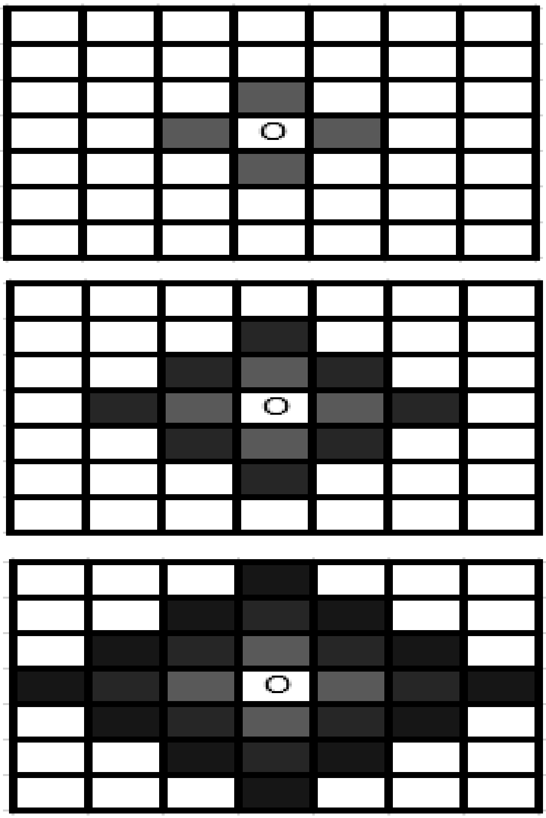

Fig. 1: (a, b and c) Geometry of the diamond patterns.

A recurrence relation $a_r = 4r + a_{r-1}$ is observed in calculation of total number of cells a_r covered by a diamond of radius r. In this relation, the term a_{r-1} is the number of cells covered by a diamond of radius r-1 and $a_0 = 1$. The solution of this linear, non-homogeneous recurrence relation with constant coefficients is given as:

$$a_r = 2r^2 + 2r + 1 \tag{1}$$

This equation provides the total number of cells in a diamond of radius r. According to the DDP, a symbol that follows a dispersion size r, leaves a_r number of cells around it before repeating itself.

DDP is committed to disperse symbols in a Latin square. We can confirm this pattern by drawing a diamond around any symbol. The symbol in the center of the diamond does not appear on other locations of the same diamond. All the same symbols are dispersed at a Manhattan distance equivalent to the dispersion size of the Latin square. This distance is also equal to the radius of the diamond.

An increasing dispersion size requires more symbols to hold cells in between same symbols. This results in the formation of a higher order Latin square. The following formula calculates the required number of symbols n (order of Latin square) for a dispersion r [5].

$$n = \frac{r^2+r}{2} + \frac{r}{2} + 1 \tag{2}$$

The above equation uses integral division. To avoid integral division, an equivalent formula is given as:

$$n = \frac{1}{4}(2r^2 + 4r + (-1)r + 3) \tag{3}$$

Table 1 shows the values for number of cells and number of symbols required for each dispersion size r.

The term $2r^2 + 4r + (-1)r + 3$ has significant meanings noticed in third column. All these numbers are multiples of four. The difference between two terms follows a pattern shown in column 5 and 6.

Table 1: Number of cells and required number of symbols in diamond with radius r.

Dispersion Size r	Cells in the Diamond $2r^2+2r+1$	$t(r)= 2r^2+4r+(-1)^r+3$	Symbols Required n=t/4	Difference D(t)=t(r)-t(r-1)	D(t)-D(t-2)
1	5	8	2		
2	13	20	5	12	
3	25	32	8	12	
4	41	52	13	20	8
5	61	72	18	20	
6	85	100	25	28	8
7	113	128	32	28	
8	145	164	41	36	8

As discussed in coming section, such span of the propagation is quite relevant with the physical structure of a diamond that is symmetrical in all its four parts (Fig. 2).

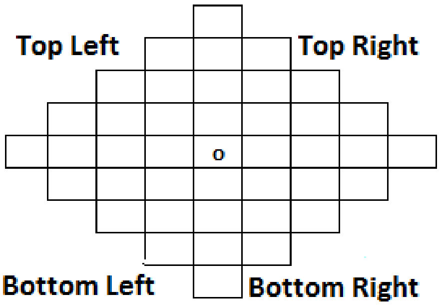

Fig. 2: Four symmetrical parts of a diamond.

A dispersion size of two is possible only for a Latin square that has at least five symbols (Table 1). Fig. 3(a) shows a complete Latin square formed by only five available symbols. Similarly, Figs. 3(b) and 3(c) show all the required symbols spread over different cells for dispersion sizes three and four, respectively.

b	d	c	a	e
a	e	b	d	c
d	c	a	e	b
e	b	d	c	a
c	a	e	b	d

(a)

a	e	f	d	g	b	h	c
d	g	b	h	c	a	e	f
h	c	a	e	f	d	g	b
e	f	d	g	b	h	c	a
g	b	h	c	a	e	f	d
c	a	e	f	d	g	b	h
f	d	g	b	h	c	a	e
b	h	c	a	e	f	d	g

(b)

a	e	j	f	k	d	l	g	b	h	m	i	c
d	l	g	b	h	m	i	c	a	e	j	f	k
m	i	c	a	e	j	f	k	d	l	g	b	h
j	f	k	d	l	g	b	h	m	i	c	a	e
g	b	h	m	i	c	a	e	j	f	k	d	l
c	a	e	j	f	k	d	l	g	b	h	m	i
k	d	l	g	b	h	m	i	c	a	e	j	f
h	m	i	c	a	e	j	f	k	d	l	g	b
e	j	f	k	d	l	g	b	h	m	i	c	a
l	g	b	h	m	i	c	a	e	j	f	k	d
i	c	a	e	j	f	k	d	l	g	b	h	m
f	k	d	l	g	b	h	m	i	c	a	e	j
b	h	m	i	c	a	e	j	f	k	d	l	g

(c)

Fig. 3: (a, b and c) Latin squares of dispersion size 2, 3 and 4.

3. Results

Diamond dispersion pattern is a well-defined pattern that exhibits many interesting attributes. Followings are some of our major findings about diamond dispersed pattern in Latin squares.

3.1 Patterns of Repetition

Fig. 4(a) is a diamond with dispersion size three. A few highlighted cells follow a certain pattern. Each of the symbols in these highlighted cells repeats four times and among the remaining symbols f &h repeat three times. The symbol g repeats two times only. A symbol can repeat maximum four times. As all the symbols compete for available cells/locations, some symbols do not get the chance of repeating themselves four times. A unique pattern among symbols that are repeated four times (highlighted) for a dispersion size of four can be noticed in Fig. 4(b).

In Table 1, the value of n for odd dispersions has a sequence of numbers (2, 8, 18, …). The general term of this progression is $2p^2$. Similarly, the order of LS for all even dispersion follows a sequence (5, 13, 25, …). The general term of this sequence is given by:

$$2p^2 + 2p + 1 \tag{4}$$

Instead of using Eq. 2 or Eq. 3, we can use these individual and simple equations for odd and even dispersions.

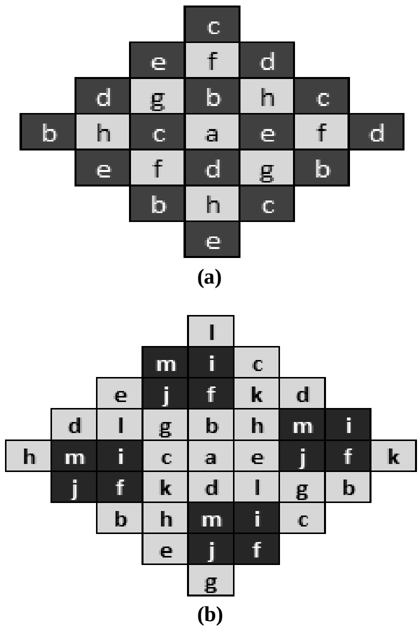

Fig. 4: (a, b) Repetition patterns.

3.2 Almost a Quarter of the Diamond Need Unique Symbols

A diamond of radius four is shown in Fig. 5. The highlighted cells (quarter of the diamond) form a smaller diamond of radius two consisting of total thirteen cells. As a symbol cannot repeat more than four times, a quarter of total cells must hold unique symbols. We need thirteen unique symbols to fill out this diamond.

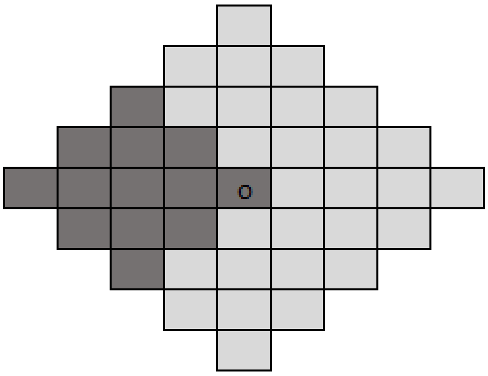

Fig. 5: Diamond of Radius r and its quadrant.

The relation of symbols and cells for all even dispersions is presented in Table 2. This relation is also evident from equation (1) and equation (4) as discussed above.

Table 2: Relation between diamonds of radius r and its quadrant.

M.D for diamond of radius r (Even)	2	4	6	8	10
Symbols required	5	13	25	41	61
Cells in diamond of radius r/2 cells = $2(r/2)^2 + 2(r/2) + 1$	5	13	25	41	61

3.3 *Cells to Symbols Ratio*

To disperse same symbols over a long distance requires adding more cells and symbols. By increasing number of symbols and cells in a Latin square, the size of resulting diamond also increases. Cells to symbols ratio is helpful to optimize the size of diamond.

Cells-to-symbols ratio for different dispersion sizes is given in Table 3. As central symbol always takes one central block and never repeats in other parts of the diamond, we exclude this in the calculations. The ratio of (cells-1) to (symbols-1) represents number of cells covered by a single symbol. As discussed above, this value is around 4 normally.

Table 3: Cells to symbols ratio.

Dispersion Size	2	3	4	5	6	- - - -	50
Cells-1	12	24	40	60	84	- - - -	5100
Symbols-1	4	7	12	17	24	- - - -	1300
Cells/Symbols	3.0	3.4	3.3	3.5	3.5	- - - -	3.9

We may notice that the value starts from 3 and for higher dispersion sizes, this value approximately reaches up-to 4. The presence of crevices (see next attribute) inside a diamond make this value less than 4.

3.4 *Repeating Pattern in DDP*

A diamond of radius eleven with the center O1 is shown in Fig. 6. All four quarters are marked with center O2. They have radius five and they are further divided into four of their quadrants with centers marked as O3. Similar patterns are recurring at smaller scales progressively.

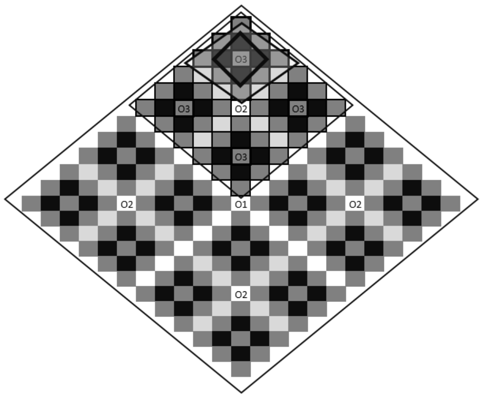

Fig. 6: Repeating patterns in DDP.

There are some light cells (unshaded) that are not a part of any quadrant; we call these cells crevices of the diamond. These crevices cause extra symbols to be introduced for a given dispersion size and consequently increase the order of a resultant Latin square. The presence of crevices reduces cells to symbols ratio.

3.5 *KV Squares of DDP*

We devised an equation to generate Latin squares of dispersed pattern. This pattern in Latin square is a more generalized form of KV Squares. The following equation of DDP produces a Latin square of order n with a dispersion size r. Here S(i, j) represents the resultant symbol S to be assigned to the cell at row i and column j.

$$s(i,j) = [(r + 1 - r\%2)i + j]\%n \qquad (5)$$

The numbers (r+1) and n are mutually prime and 0 < (r+1) < n (Table 4). These two conditions are the main requirements for generating a KV square.

Table 4: Co-prime relation of r + 1 and n.

r + 1	3	5	7	9	11
n	5	13	25	41	61

3.6 *Randomization Space*

In DDP, there may exist multiple Latin squares of same order and same dispersion size. This characteristic of DDP is very useful when an experimental design needs variety of implementation styles. By using equation 5, we draw different clones of a LS with a dispersion size 2 (Fig. 7). These clones are just different permutations of a Latin square. As the number of Latin squares increases with increasing order exponentially, the randomization space for DDP clones also outgrow with higher dispersion sizes.

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

1	2	3	4	0
4	0	1	2	3
2	3	4	0	1
0	1	2	3	4
3	4	0	1	2

2	3	4	0	1
0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3

0	3	1	4	2
1	4	2	0	3
2	0	3	1	4
3	1	4	2	0
4	2	0	3	1

1	4	2	0	3
2	0	3	1	4
3	1	4	2	0
4	2	0	3	1
0	3	1	4	2

2	0	3	1	4
3	1	4	2	0
4	2	0	3	1
0	3	1	4	2
1	4	2	0	3

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

1	2	3	4	0
3	4	0	1	2
0	1	2	3	4
2	3	4	0	1
4	0	1	2	3

2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2
0	1	2	3	4

Fig. 7: Clones of LS with DDP.

3.7 *DDP as a Solution of Arrangement Problems*

Other than its main use as design of experiments, DDP is useful in some routine life practices as solution of arrangement problems. Given below are a few examples of arrangement problems that may be solved using DDP.

3.7.1 *Scattering image/video data to help concealment by decoder*

Spatial and temporal redundancy is exploited to compress a video sequence for communication. On receiving side, the decoder may not receive all the transmitted packets of video data; in such situation, concealment is required for recovery. It is very much preferred to receive data adjacent to a missing part of image frame that can overcome concealment artifacts. DDP is useful in scattering the spatial information available in an image frame during its compression stage. Even if consecutive packets of data are lost due to a bad communication channel, it is still possible to receive neighboring blocks of a missing block in an image frame. These available neighboring blocks may help to conceal better and to improve subjective and objective quality of a decoded video frame.

An example of an erroneous video frame that is concealed at decoder side is shown in Fig. 8. As blocks of video frame were dispersed during compression stage, all the missing blocks are not concentrated at one location that make concealment job easy and more effective [10].

Fig. 8: Dispersed pattern implemented in H.264/AVC.

3.7.2 *Segregation of incompatible chemicals*

In a storage place of different chemicals, there is an issue for segregating incompatible chemicals. Incompatible chemicals are dangerous when mixed or put together. They must set apart from each other. There is available a complete classification of chemicals with respect to their incompatibility. Unavailability of enough space may limit to put incompatible chemicals in the same cabinet. DDP is useful in allocation of incompatible chemicals to different boxes and shelves of a cabinet.

Suppose there are seven boxes in each shelf of a cabinet with total six shelves. If we have a set of incompatible chemicals (C1, C2, …. C9) to be segregated in this cabinet, an arrangement can use DDP of five symbols in a 7×6 gird of cells with a dispersion size of 2 (Fig. 9). Each cell represents a box and all incompatible chemicals from a set may use locations of a symbol in the grid. For example, symbol 0 is replaced (highlighted) by all incompatible chemicals of the given set. Similarly, the locations of symbol 1 can provide space for incompatible chemicals from another set.

C1	1	2	3	4	C2	1
3	4	C3	1	2	3	4
1	2	3	4	C4	1	2
4	C5	1	2	3	4	C6
2	3	4	C7	1	2	3
C8	1	2	3	4	C9	1

Fig. 9: Segregation of incompatible chemicals.

3.7.3 *Separation of students to reduce examination malpractices*

To reduce the incidents of cheating in examination, several methods are adopted [11, 12]. Some of them try to overcome the issue of cheating in examination through collusion by examinees who are seated at adjacent seats. Separation of students becomes mandatory when many students are tested at one time. In normal practices, students attempting different exams are mixed in a big hall.

DDP is useful in curbing cheating by confirming separation of students attempting the same examination. In such environment, DDP is plotted for a gird of available number of seats (column) and available lines (rows), where number of symbols are equivalent to types of examinations held in the hall. The general equation (5) is used to arrange five different examinations (E1, E2, E3, E4 and E5) conducted in a hall where twelve seats are available in each of total eight lines (Fig. 10). In such arrangements, no two students conducting same examination become a direct neighbor to each other.

E1	E2	E3	E4	E5	E1	E2	E3	E4	E5	E1	E2
E4	E5	E1	E2	E3	E4	E5	E1	E2	E3	E4	E5
E2	E3	E4	E5	E1	E2	E3	E4	E5	E1	E2	E3
E5	E1	E2	E3	E4	E5	E1	E2	E3	E4	E5	E1
E3	E4	E5	E1	E2	E3	E4	E5	E1	E2	E3	E4
E1	E2	E3	E4	E5	E1	E2	E3	E4	E5	E1	E2
E4	E5	E1	E2	E3	E4	E5	E1	E2	E3	E4	E5
E2	E3	E4	E5	E1	E2	E3	E4	E5	E1	E2	E3

Fig. 10: Examinees sitting arrangements using DDP.

4. Conclusions

The proposed pattern in Latin squares provides a complete arrangement system for treatments in experimental designs. The current study focuses to analyze the system of DDP. It includes discussion on the physical layout of DDP and its attributes. DDP is recommended for the experimental designs where treatments are supposed to be dispersed or not taking place quite soon. Except experimental designs for statistical analysis, DDP is suggested for use in arrangement problems of routine life or other scientific disciplines.

References

[1] B. Durakovic, "Design of experiments application, concepts, examples: State of the art", PEN, vol. 5, no. 3, p. 421-439, 2017.

[2] S. Aoki and A. Takemura, "Design and analysis of fractional factorial experiments from the viewpoint of computational algebraic statistics", J. Stat. Theory Pract., vol. 6, pp. 147-161, 2012. https://doi.org/10.1080/15598608.2012.647556.

[3] D.R. Cox, "A remark on systematic Latin squares", J. R . Stat. Soc., vol. 41, no. 3, pp. 388-389, 1979, https://doi.org/10.1111/j.2517-6161.1979.tb01093.x.

[4] A. Hedayat, "A complete solution to the existence and non-existence of Knut Vik designs and orthogonal Knut Vik designs", J. Comb. Theory, vol. 22, pp. 331-337, 1978.

[5] S.Z. Mahfooz and M. Akhlaq, "Latin squares of diamond dispersed pattern", Int. J. Appl. Comput. Math., vol. 3, pp. 81-89, 2017.

[6] R.A. Fisher, "The arrangement of field experiments", J. Mini. Agri. UK, vol. 33, pp. 503-513, 1926.

[7] F.G. Giesbrecht and M.L. Gumpertz, "Planning, construction and statistical analysis of comparative experiments", John Wiley & Sons, Inc., April 2004.

[8] U. Saint-Mont, "Randomization does not help much, comparability does", PloS, vol. 10, no. 7, 2015, https://doi.org/10.1371/journal.pone.0132102.

[9] V. Dabbaghian and Tiankuang Wu, "Constructing non-cyclic pandiagonal Latin squares of prime orders", J. Discrete Algorithms, vol. 30, pp. 70-77, 2015.

[10] S.Z. Mahfooz and Bo Hu, "Diamond dispersed pattern in FMO for error resilient H.264/AVC video communication", J. Mianyang Normal Univ, vol. 11, pp. 101-106, 2011.

[11] M. Abdaoui, "Strategies for avoiding cheating and preserving academic integrity in tests", Alkhitab w el-Tawassol Journal, vol. 4, pp. 1-9, 2018.

[12] J. Ramberg and B. Modin, "School effectiveness and student cheating: Do students' grades and moral standards matter for this relationship?", Soc. Psychol. Educ, vol. 22, pp. 517-538, 2019.