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Efficient Regular Graph Generalized Neighbor Designs

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ABSTRACT

In this study we have constructed a class of economical block designs called regular graph generalized neighbor designs for circular blocks of size 4 and v ≤ 50. This class of design can provide an efficient alternative of universally optimal neighbor designs. These designs will increase the applications of regular graph designs particularly in the fields of serology, agro forestry, agriculture, etc. In terms of efficiency factor, proposed designs are more efficient as compared with other generalized neighbor designs. The two best designs, for v =26 and v =50, have percentage of upper bound 99.91% and 99.93% respectively.

Keywords: *Binary designs, Regular graph designs, GNDs, Efficiency factor of GN_2-designs*

1. Introduction

A regular graph generalized neighbor designs (RGGND) is a design which attains the properties of both regular graph designs (RGDs) and generalized neighbor designs (GNDs). Conditions of treatment balanced and neighbor balanced are dually relaxed. RGD is a particular class of partially balanced incomplete block design with two associate scheme {PBIBD (2)} in the same block every treatment pairs occurs either λ_1 or $\lambda_2 = \lambda_1+1$ times. RGDs are, therefore, considered much closer to the balanced incomplete block designs [BIBDs]. On the other hand, GNDs were introduced by Misra et al. [1], each ordered pair of treatment appears λ_i(1,2… t) times as neighbor is called GN_t-design. If λ_i takes only two values as λ_1 and λ_2 then these are called GN_2-designs. GN_2-designs will be minimal if $\lambda_1 = 1$ and $\lambda_2 = 2$.

RGGNDs (i) belong to the PBIBDs with two associate schemes, and (ii) are nearly balanced for neighbor effect. These designs can be applied in all practical fields like agriculture, biological sciences and agro forestry where incomplete block designs and neighbor designs or both are applicable. Specifically, these designs are applied in *(i)* biological sciences where antigens/viruses are arranged in circular plates, and (ii) plant breeding experiments, where varieties are effected by difference in height, root vigor, plant position, germination date, etc. Considerable amount of experimental material can be saved by sacrificing a very small proportion of efficiency. GNDs were addressed by several authors particularly [2-4]; while [5-9] have constructed some GN_2 and GN_3-designs. For a detailed review of RGDs, one is referred to Cakiroglu [10] and the references therein. This paper comprises construction of Regular graph (RG) GN_2-designs along with their efficiencies for blocks of size *4*. The remaining part of this paper is organized as follows: Section *2* describes the model for the analysis of RG GN_2-designs. Construction procedure of proposed designs is elaborated in Section 3. Section 4 presents the catalogue of RG GN_2-designs. Comparison of existing and new designs is given in sections 5, whereas, section 6 includes the discussion of the paper.

2. The Model

Following model is proposed by Misra et al. [1], for analysis of NDs.

$$y_{ij} = \mu + \tau_{d(i,j)} + \alpha_{d(i,i+1)} + \alpha_{d(i,i-1)} + \beta_j + \varepsilon_{ij}$$

$$i = 1, 2, \ldots, v \text{ and } j = 1, 2, \ldots, b \qquad (1)$$

Iqbal et al. [11] recommended this model as:

$$y = X_0\mu + X_1\alpha + X_2\alpha + X_3\beta + \varepsilon \qquad (2)$$

Where X_0 and y are observations vectors, *1's* each of order *(bk×1)* respectively, X_1 (for treatment effects) and X_2 (for neighbor effects) are incidence matrices of order *(bk×v)*. X_3 (for block effects) is incidence matrix of order *(bk×b)*, where k is equal block size. Model *(2)* is partitioned from general linear model in matrix form, i.e., $y = XP + \varepsilon$ Matrix X and vector P are partitioned for model *(2)* as:

$$X = [X_0 : X_1 : X_2 : X_3] \text{ and } P' = [\mu' : \tau' : \alpha' : \beta']$$

Then the information matrix for model (2) is accordingly.

$$X'X = \begin{bmatrix} X_0'X_0 & X_0'X_1 & X_0'X_2 & X_0'X_3 \\ X_1'X_0 & X_1'X_1 & X_1'X_2 & X_1'X_3 \\ X_2'X_0 & X_2'X_1 & X_2'X_2 & X_2'X_3 \\ X_3'X_0 & X_3'X_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}$$

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$$= \begin{bmatrix} X_0'X_0 & X_0'X_1 & X_0'X_2 & X_0'X_3 \\ X_1'X_0 & X_1'X_1 & L & N \\ X_2'X_0 & L' & X_2'X_2 & 2N \\ X_3'X_0 & N' & 2N' & X_3'X_3 \end{bmatrix}$$

$X'_1 X_2 = L$, the element (i,i') of matrix L is the number of times treatment i' appears in the neighbor (either left or right) of treatment i where $i' \neq i$; and $X'_1 X_3 = N$ is the incidence matrix. Then the matrix NN' is a concurrence matrix for treatment effect whose $(i,i')^{th}$ entry is the number that treatment i and i' occurs together in same block(s) where $i' \neq i$.

For a neighbor balanced design, all off-diagonal elements of matrix L must be the same. If its off-diagonal elements contain only two distinct values, the design is known as GN_2-design. Similarly, if off-diagonal elements of concurrence matrix NN' are same then the design is BIBD otherwise PBIBD. Specifically if it takes two different values (λ_1 and $\lambda_2 = \lambda_1 + 1$) then the design is RG. Thus, in order to construct RG GN_2-design, off-diagonal elements of matrix L should be only two different values and off-diagonal elements of concurrence matrix NN' must be two different values with at most difference one.

3. Method of Cyclic Shifts (Construction Method)

Method of cyclic shifts is explained here for the construction of RGDs, GNDs, and RG GN_2–designs. For further detail, see Iqbal et al. [11].

Let $Q_m = [q_{m1}, q_{m2}, \ldots, q_{m(k-1)}]$; $1 \leq q_{mj} \leq v-1$, $(q_{m1} + q_{m2} + \ldots + q_{m(k-1)}) \neq 0 \ (mod\ v)$, $m = 1, 2, \ldots, c$ be the m^{th} set of shifts, c is the number of sets of shifts requisite for v and k. The IB for a design constructed from every Q_m is $I_m = (0, q_{m1}, (q_{m1}+q_{m2}), \ldots, (q_{m1}+\ldots+q_{m(k-1)})) \ (mod\ v)$.

For a binary blocks design, aggregate of any two, three … or $(k-1)$ successive elements of Q_m should not be 0 (mod v), if so reorder the elements. For the Construction of Treatment Balanced and NDs, Q^*contains:

i. Each q_{mj} and v-q_{mj},
ii. Sum (mod v) of all elements of every Q_m along with its complement.
iii. Sum (mod v) of succeeding two, three, ..., $(k-2)$ units of every Q_m along with their complements.

3.1 Neighbor Balanced and GND

if Q^* based on conditions (i) and (ii) and consists of $1, 2, \ldots, (v-1)$ an equal number of times, say λ', is called NBD but if the values Q^* has unequal nonzero frequencies of $1, 2, \ldots, (v-1)$, say λ'_i and takes t different values then it is called GN_t-design. If λ'_i takes only two values, then it is called GN_2-design.

3.2 RGD

In addition of condition *(iii)* along with conditions *(i)* and *(ii)* in Q^*, if Q^* contains $1, 2, \ldots, (v-1)$ an equal number of times, say λ, the design is called BIBD but if the values Q^* has unequal nonzero frequencies of $1, 2, \ldots, (v-1)$, say λ_i, then design is called PBIBD. If λ_i takes on λ_1 and $\lambda_2 = \lambda_1 + 1$ two values only, then it is called RGD. ND will be universally optimal if it is BIBD as well.

3.3 RG GN_2-design

A design is called RG GN_2-design if λ_i takes only two values λ'_1 and $\lambda'_2 = \lambda'_1 + 1$ and λ_i takes only two values λ_1 and $\lambda_2 = \lambda_1 + 1$.

3.4 Efficiency Factor and Upper Bound

Hinkelmann and Kempthorne [12] suggested a method to compare an incomplete block design with complete block design or with other incomplete block design. In this study relative efficiency factor *(E)* of design are computed with respect to complete block designs. Another measure known as upper bound for efficiency factor *(UB)* is computed for these designs which show maximum efficiency of given incomplete block design. A BIBD always achieves its upper bound, i.e., $E = UB$.

Example : Consider a design with $v = 6$ and $k = 4$, lets take a set of shifts $Q = \{4,3,1\}$ with corresponding Initial Block $(IB) = \{0,4,1,2\}$. The complete design is:

IB	Other Blocks				
1	2	3	4	5	6
0	1	2	3	4	5
4	5	0	1	2	3
1	2	3	4	5	0
2	3	4	5	0	1

Here $m = 1$ (only one set of shifts) and $j = 1, 2, 3\ (= k-1)$. The elements under conditions *(i)*, *(ii)* and *(iii)* for this design are $\{4,3,1,2,3,5\}$, $\{2, 4\}$ and $\{1,4, 5,2\}$ respectively.

Q^* (condition *(i)* and *(ii)*); $\{4,3,1,2,3,5,2,4\}$. Treatments 1 to $v-1 = 5$ appear either once or twice, i.e. $\lambda'_1 = 1$ and $\lambda'_2 = 2$.

Q^* (condition *(i)*, *(ii)* and *(iii)*): $\{4,3,1,2,3,5,2,4,1,4,5,2\}$, Treatments 1 to $v-1 = 5$ appear either two or three times, *i.e.* $\lambda_1 = 2$ and $\lambda_2 = 3$. As λ'_i takes two values and λ_i also takes two values, therefore it is RG GN_2-design.

For an existing design, above properties can be investigated through the following calculations from model *(2)*:

$$X_1'X_3 = N = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$X_1'X_2 = L = \begin{bmatrix} 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$$

$$NN' = \begin{bmatrix} 4 & 2 & 3 & 2 & 3 & 2 \\ 2 & 4 & 2 & 3 & 2 & 3 \\ 3 & 2 & 4 & 2 & 3 & 2 \\ 2 & 3 & 2 & 4 & 2 & 3 \\ 3 & 2 & 3 & 2 & 4 & 2 \\ 2 & 3 & 2 & 3 & 2 & 4 \end{bmatrix}$$

Since the off-diagonal elements of matrix L are either *1* or *2* so the above is GN_2-design with $\lambda'_1 = 1$ and $\lambda'_2 = 2$. Similarly, matrix *NN'* has either *2* or *3* off-diagonal elements, so the design is RG. Hence the above design is RG GN_2-design.

4. RG GN2-designs Construction

Twenty-nine regular graph GN_2-designs for $k = 4$ were constructed and $12 \leq v \leq 50$ by using method of cyclic shifts. These designs are binary and have the property of connectedness. The new constructed designs were shown in Table 1 of appendix.

5. Discussion

In this paper, we have constructed a class of economical designs called, RGGNDs specifically, RG GN_2-design. The main objective to introduce this class is to provide an efficient alternative of universally optimal NDs. 29 new RG GN_2-designs are constructed for block size 4 and are compiled in Table 1 of appendix. We have identified GN_2-designs with block size 4 from review of literature. Previously, researchers have developed some series or catalogue of GN_2-designs [4, 6-8]. Most GN_2 designs are PBIBD *(3, 4 or 5)*; very few are PBIBD *(2)*, and one design of *(6)* for $v = 7$ is BIBD. Efficiency factor *(E)*, upper bound and percentage of upper bound were calculated for newly proposed and existing designs. The results of investigation of existing designs are summarized in Tables 2 to 5 of Appendix, whereas, Table 6 contains comparison of proposed designs and existing designs.

6. Conclusions

The proposed 29 RG GN_2-designs, for circular blocks of size 4 and v ≤ 50, are given in appendix Table 1 with initial blocks and other parameters. These designs have possibly less association scheme i.e., PBIBD *(2)* as well as RGD. Efficiency factors *(E)* in Table 6 show that these designs are not only more efficient as compared with existing designs on the basis of associate scheme and efficiency factor but also closest to BIBD. The percentage of upper bound for two designs (for $v = 26$ and $v = 50$) are as high as 99.91% and 99.93%, respectively. The improved treatment balanced property of new GN_2-designs has enhanced their applications and validity.

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Appendix

Table 1: Construction of RG GN_2-designs for k = 4.

v	Initial Block(s)	GN_2-Designs		RGD	
		n'_1, n'_2	λ'_1, λ'_2	n(s)	λ(s)
12	(0,11,2,6) (0,8,1,3)	6,5	1,2	9,2	2,3
14	(0,13,3,5) (0,3,10,2)	10,3	1,2	2,11	1,2
15	(0,14,3,5) (0,3,10,1)	12,2	1,2	4,10	1,2
20	(0,1,3,6) (0,4,9,16) (0,8,18,7)	14,5	1,2	2,17	1,2
21	(0,1,3,6) (0,4,9,16) (0,8,18,7)	16,4	1,2	4,16	1,2
22	(0,1,3,6) (0,4,9,16) (0,9,20,8)	18,3	1,2	6,15	1,2
23	(0,1,6,15) (0,4,16,3) (0,21,4,11)	20,2	1,2	8,14	1,2
24	(0,1,3,6) (0,7,11,16) (0,10,22,9)	22,1	1,2	10,13	1,2
26	(0,2,3,8) (0,3,9,13) (0,7,15,19) (0,17,2,12)	18,7	1,2	2,23	1,2
27	(0,26,2,7) (0,2,10,14) (0,21,1,10) (0,11,23,14)	20,6	1,2	4,22	1,2
28	(0,1,6,12) (0,2,5,9) (0,7,15,25) (0,11,26,12)	22,5	1,2	6,21	1,2
29	(0,1,4,6) (0,4,9,16) (0,8,18,27) (0,11,23,8)	24,4	1,2	8,20	1,2
30	(0,1,5,7) (0,5,8,14) (0,13,27,12) (0,20,11,22)	26,3	1,2	10,19	1,2
32	(0,1,3,6) (0,4,9,17) (0,22,2,16) (0,25,4,13)	30,1	1,2	14,17	1,2
34	(0,1,8,11) (0,2,11,15) (0,5,18,24) (0,8,25,3) (0,18,32,4)	26,7	1,2	6,27	1,2
35	(0,1,4,9) (0,2,6,12) (0,7,15,26) (0,10,23,3) (0,14,30,13)	28,6	1,2	8,26	1,2
36	(0,1,3,7) (0,5,13,31) (0,6,15,25) (0,12,25,3) (0,15,32,16)	30,5	1,2	10,25	1,2
37	(0,1,5,15) (0,8,10,23) (0,3,8,19) (0,6,18,34) (0,7,24,33)	32,4	1,2	12,24	1,2
38	(0,3,4,13) (0,4,6,16) (0,8,13,31) (0,32,5,24) (0,12,35,14)	34,3	1,2	14,23	1,2
39	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,11,28,12) (0,15,34,13)	36,2	1,2	16,22	1,2
40	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,11,28,12) (0,15,34,13)	38,1	1,2	18,21	1,2
42	(0,1,4,12) (0,2,6,15) (0,5,15,29) (0,6,22,41) (0,17,37,26) (0,17,37,26)	34,7	1,2	10,31	1,2
43	(0,1,3,6) (0,4,11,19) (0,5,16,25) (0,15,21,1) (0,17,30,9) (0,33,2,19)	36,6	1,2	12,30	1,2
44	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,33,11,24) (0,17,3,24) (0,15,43,17)	38,5	1,2	14,29	1,2
45	(0,1,3,6) (0,4,11,19) (0,5,14,25) (0,13,28,1) (0,22,39,18) (0,35,2,16)	40,4	1,2	16,28	1,2
46	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,35,3,16) (0,31,7,30) (0,29,2,20)	42,3	1,2	18,27	1,2
47	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,36,2,16) (0,32,3,28) (0,21,40,16)	44,2	1,2	20,26	1,2
48	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,36,2,16) (0,32,3,28) (0,21,40,16)	46,1	1,2	22,25	1,2
50	(0,1,4,9) (0,2,6,12) (0,7,15,25) (0,11,24,38) (0,35,1,18) (0,31,2,24) (0,30,3,19)	42,7	1,2	14,35	1,2

Table 2: Examination of GN_2-designs by Shehzad et al. [8].

V	T	W	Initial Block(s)	GN_2- Designs		PBIBD Designs		E	UB
				$n'_1, n'_2,$	$\lambda'_1, \lambda'_2,$	$n(s)$	$\lambda(s)$		
Series 2.1									
10	1	0	(0,9,1,4)(0,5,9,2)	2,7	1,2	2,3,4	4,4,1	0.8289	0.8333
11		1	(0,10,1,4)(0,6,10,2) *	4,6	1,2	6,4	2,3	0.8218	0.8250
18	2	0	(0,17,1,4)(0,13,1,8)(0,9,17,6)	10,7	1,2	4,9,2,2	1,2,3,4	0.7869	0.7941
19		1	(0,18,1,4)(0,14,1,8)(0,10,18,6)	12,6	1,2	6,8,2,2	1,2,3,4	0.7817	0.7917
26	3	0	(0,25,1,4)(0,21,1,8)(0,17,1,12) (0,13,25,10)	18,7	1,2	12,8,2,1,2	1,2,3,4,5	0.7678	0.7800
27		1	(0,26,1,4)(0,22,1,8)(0,18,1,12) (0,14,26,10)	20,6	1,2	10,14,2	1,2,5	0.7697	0.7788
34	4	0	(0,33,1,4)(0,29,1,8)(0,25,1,12) (0,21,1,16)(0,17,33,14)	26,7	1,2	18,9,4,2	1,2,3,6	0.7610	0.7727
35		1	(0,34,1,4)(0,30,1,8)(0,26,1,12) (0,22,1,16)(0,18,34,14)	28,6	1,2	18,12,2,2	1,2,3,6	0.7610	0.7721
42	5	0	(0,41,1,4)(0,37,1,8)(0,33,1,12) (0,29,1,16)(0,25,1,20)(0,21,41,18)	34,7	1,2	26,8,4,1,2	1,2,3,4,7	0.7562	0.7683
43		1	(0,42,1,4)(0,38,1,8)(0,34,1,12) (0,30,1,16)(0,26,1,20)(0,22,42,18)	36,6	1,2	22,18,2	1,2,7	0.7576	0.7679
50	6	0	(0,49,1,4)(0,45,1,8)(0,41,1,12) (0,37,1,16)(0,33,1,20)(0,29,1,24) (0,25,49,22)	42,7	1,2	32,9,6,2	1,2,3,8	0.7535	0.7653
51		1	(0,50,1,4)(0,46,1,8)(0,42,1,12) (0,38,1,16)(0,34,1,20)(0,28,1,24) (0,26,50,22)	44,6	1,2	30,16,2,2	1,2,3,8	0.7535	0.7650
Series 2.2									
12	2	0	(0,11,1,4)(0,7,1,2)	6,5	1,2	4,3,2,2	1,2,3,4	0.7992	0.8182
13		1	(0,12,1,4)(0,8,1,2)	8,4	1,2	4,6,2	1,2,3	0.7957	0.8125
20	3	0	(0,19,1,4)(0,15,1,8)(0,11,1,2)	14,5	1,2	8,9,2	1,2,5	0.7745	0.7895
21		1	(0,20,1,4)(0,16,1,8)(0,12,1,2)	16,4	1,2	10,8,2	1,2,4	0.7716	0.7875
28	4	0	(0,27,1,4)(0,23,1,8)(0,19,1,12) (0,15,1,2)	22,5	1,2	16,7,2,2	1,2,3,6	0.7613	0.7778
29		1	(0,28,1,4)(0,24,1,8)(0,20,1,12) (0,16,1,2)	24,4	1,2	16,10,2	1,2,5	0.7613	0.7768
36	5	0	(0,35,1,4)(0,31,1,8)(0,27,1,12) (0,23,1,16)(0,19,1,2)	30,5	1,2	20,13,2	1,2,7	0.7564	0.7714
37		1	(0,36,1,4)(0,32,1,8)(0,28,1,12) (0,24,1,16)(0,20,1,2)	32,4	1,2	22,12,2	1,2,6	0.7564	0.7708
44	6	0	(0,43,1,4)(0,39,1,8)(0,35,1,12) (0,31,1,16)(0,27,1,20)(0,23,1,2)	38,5	1,2	28,11,2,2	1,2,3,8	0.7521	0.7674
45		1	(0,44,1,4)(0,40,1,8)(0,36,1,12) (0,32,1,16)(0,28,1,20)(0,24,1,2)	40,4	1,2	28,14,2	1,2,7	0.7535	0.767
Series 2.3									
6	1	0	(0,4,1,2)*	2,3	1,2	3,2	2,3	0.8929	0.9000
7		1	(0,5,1,2)**	4,2	1,2	6	2	0.8750	0.8750
14	2	0	(0,13,1,4)(0,8,1,6)	10,3	1,2	4,7,2	1,2,3	0.7997	0.8077
15		1	(0,14,1,4)(0,9,1,6)	12,2	1,2	6,6,2	1,2,3	0.7952	0.8036
22	3	0	(0,21,1,4)(0,17,1,8)(0,12,1,10)	18,3	1,2	10,9,2	1,2,4	0.7759	0.7857

23		1	(0,22,1,4)(0,18,1,8)(0,13,1,10)	20,2	1,2	14,4,2,2	1,2,3,4	0.7723	0.7841
30	4	0	(0,29,1,4)(0,25,1,8)(0,21,1,12) (0,16,1,14)	26,3	1,2	16,11,2	1,2,5	0.7650	0.7759
31		1	(0,30,1,4)(0,26,1,8)(0,22,1,12) (0,17,1,14)	28,2	1,2	18,10,2	1,2,5	0.7650	0.7750
38	5	0	(0,37,1,4)(0,33,1,8)(0,29,1,12) (0,25,1,16) (0,20,1,18)	34,3	1,2	22,13,2	1,2,6	0.7599	0.7703
39		1	(0,38,1,4)(0,34,1,8)(0,30,1,12) (0,26,1,16)(0,21,1,18)	36,2	1,2	26,8,2,2	1,2,3,6	0.7587	0.7697
46	6	0	(0,45,1,4)(0,41,1,8)(0,37,1,12) (0,33,1,16)(0,29,1,20)(0,24,1,22)	42,3	1,2	28,15,2	1,2,7	0.7562	0.7667
47		1	(0,46,1,4)(0,42,1,8)(0,38,1,12) (0,34,1,16)(0,30,1,20)(0,25,1,22)	44,2	1,2	30.14.2	1,2,7	0.7562	0.7663

*PBIBD (2), **BIBD

Table 3: Examination of GN_2-designs by Yab et al. [7].

V	T	Initial Block(s)	GN_2- Designs		PBIBD Designs		E	UB
			$n'_1, n'_2,$	$\lambda'_1, \lambda'_2,$	$n(s)$	$\lambda(s)$		
Theorem 3.2								
10	1	(0,9,1,4)(0,5,7,1)	2,7	1,2	4,4,1	2,3,4	0.8278	0.8333
18	2	(0,17,1,4)(0,13,1,8)(0,9,15,16)	10,7	1,2	2,11,4	1,2,3	0.7906	0.7941
26	3	(0,25,1,4)(0,21,1,8)(0,17,1,12) (0,13,15,19)	18,7	1,2	10,10,2,3	1,2,3,4	0.7726	0.7800
34	4	(0,33,1,4)(0,29,1,8)(0,25,1,12) (0,21,1,16)(0,17,19,23)	26,7	1,2	16,11,4,2	1,2,3,5	0.7645	0.7727
42	5	(0,41,1,4)(0,37,1,8)(0,33,1,12) (0,29,1,16)(0,25,1,20) (21,23,27)	34,7	1,2	24,10,4,1,2	1,2,3,4,6	0.7590	0.7683
50	6	(0,49,1,4)(0,45,1,8)(0,41,1,12) (0,37,1,16)(0,33,1,20)(29,1,24) (0,25,27,31)	42,7	1,2	30,31,6,2	1,2,3,7	0.7567	0.7653
Theorem 3.3								
11	1	(0,10,1,4)(0,5,6,8)	4,6	1,2	2,2,6	1,2,3	0.8170	0.8250
19	2	(0,18,1,4)(0,14,1,8)(0,9,10,12)	12,6	1,2	4,12,2	1,2,4	0.7847	0.7917
27	3	(0,26,1,4)(0,22,1,8)(0,18,1,12) (0,13,14,16)	20,6	1,2	12,10,2,2	1,2,3,5	0.7678	0.7788
35	4	(0,34,1,4)(0,30,1,8)(0,26,1,12) (0,22,1,16)(0,17,18,20)	28,6	1,2	16,16,2	1,2,6	0.7622	0.7721
43	5	(0,42,1,4)(0,38,1,8)(0,34,1,12) (0,30,1,16)(0,26,1,20)(21,22,24)	36,6	1,2	24,14,2,2	1,2,3,7	0.7562	0.7679
Theorem 3.4								
12	1	(0,11,1,4)(0,5,11,1)	6,5	1,2	2,7,2	1,2,4	0.8054	0.8182
20	2	(0,19,1,4)(0,15,1,8)(0,9,19,1)	14,5	1,2	1,2,5	8,9,2	0.7745	0.7895
28	3	(0,27,1,4)(0,23,1,8)(0,19,1,12) (0,13,27,1)	22,5	1,2	14,11,2	1,2,6	0.7622	0.7778
36	4	(0,35,1,4)(0,31,1,8)(0,27,1,12) (0,23,1,16)(0,17,35,1)	30,5	1,2	20,13,2	1,2,7	0.7564	0.7714
44	5	(0,43,1,4)(0,39,1,8)(0,35,1,12) (0,31,1,16)(0,27,1,20)(21,43,1)	38,5	1,2	26,15,2	1,2,8	0.7535	0.7674
Theorem 3.5								
13	1	(0,12,2,6)(0,5,11,1)	4,6	1,2	2,8,2	1,2,3	0.8075	0.8125
21	2	(0,20,1,4)(0,16,1,8)(0,9,19,1)	16,4	1,2	10,6,2,2	1,2,3,4	0.7766	0.7875
29	3	(0,28,1,4)(0,24,1,8)(0,20,1,12) (0,13,27,1)	24,4	1,2	14,12,2	1,2,5	0.7669	0.7768
37	4	(0,36,1,4)(0,32,1,8)(0,28,1,12) (0,24,1,16)(0,17,35,1)	32,4	1,2	22,10,2,2	1,2,3,6	0.7599	0.7708
45	5	(0,44,1,4)(0,40,1,8)(0,36,1,12) (0,32,1,16)(0,28,1,20)(21,43,1)	40,4	1,2	26,16,2	1,2,7	0.7562	0.7670

Theorem 3.6								
14	1	(0,13,1,4)(0,9,1,8)	10,3	1,2	6,5,2	1,2,4	0.7886	0.8077
22	2	(0,21,1,4)(0,17,1,8)(0,13,1,12)	18,3	1,2	12,7,2	1,2,5	0.7688	0.7857
30	3	(0,29,1,4)(0,25,1,8)(0,21,1,12)(0,17,1,16)	26,3	1,2	18,9,2	1,2,6	0.7603	0.7759
38	4	(0,37,1,4)(0,33,1,8)(0,29,1,12)(0,25,1,16)(0,21,1,20)	34,3	1,2	24,11,2	1,2,7	0.7564	0.7703
46	5	(0,45,1,4)(0,41,1,8)(0,37,1,12)(0,33,1,16)(0,29,1,20)(0,25,22,23)	42,3	1,2	30,13,2	1,2,8	0.7535	0.7667
Theorem 3.7								
15	1	(0,14,1,4)(0,10,1,8)	12,2	1,2	6,6,2	1,2,3	0.7947	0.8036
23	2	(0,22,1,4)(0,18,1,8)(0,14,1,12)	20,2	1,2	12,8,2	1,2,4	0.7738	0.7841
31	3	(0,29,1,4)(0,25,1,8)(0,21,1,12)(0,17,1,16)	28,2	1,2	18,10,2	1,2,5	0.7650	0.775
39	4	(0,38,1,4)(0,34,1,8)(0,30,1,12)(0,26,1,16)(0,22,1,20)	36,2	1,2	24,12,2	1,2,6	0.7587	0.7697
47	5	(0,46,1,4)(0,42,1,8)(0,38,1,12)(0,34,1,16)(0,30,1,20)(26,1,24)	44,2	1,2	30,14,2	1,2,7	0.7562	0.7663
Theorem 3.8								
16	1	(0,15,1,4)(0,6,13,8)*	14,1	1,2	6,9	1,2	0.7952	0.8000
24	2	(0,23,1,4)(0,19,1,8)(0,15,1,12)	22,1	1,2	14,7,2	1,2,4	0.7723	0.7826
32	3	(0,31,1,4)(0,27,1,8)(0,23,1,12)(0,19,1,16)	30,1	1,2	20,9,2	1,2,5		0.7742
40	4	(0,39,1,4)(0,35,1,8) (0,31,1,12)(0,27,1,16) (0,23,1,20)	38,1	1,2	26,11,2	1,2,6	0.7587	0.7692
48	5	(0,47,1,4)(0,43,1,8)(0,39,1,12)(0,35,1,16)(0,31,1,20)(27,1,24)	46,1	1,2	32,13,2	1,2,7	0.7562	0.7660

*PBIBD(2), **BIBD

Table 4: Examination of GN_2-designs by Akhtar et al. [6].

V	Initial Block(s)	GN_2- Designs		PBIBD Designs		E	UB
		n'_1, n'_2	λ'_1, λ'_2	$n(s)$	$\lambda(s)$		
7	(0,1,3,6) **	4,2	1,2	6	2	0.8750	0.8750
10	(0,1,3,6)(0,2,7,3) *	2,7	1,2	6,3	2,4	0.8229	0.8333
11	(0,1,3,6)(0,4,10,7)	4,6	1,2	2,4,2,2	1,2,3,4	0.8112	0.8250
14	(0,1,3,6)(0,4,9,2)	10,3	1,2	6,3,4	1,2,3	0.7931	0.8077
15	(0,1,3,6)(0,4,9,1)	12,2	1,2	6,6,2	1,2,3	0.7947	0.8036
16	(0,15,1,4)(0,11,1,8)	14,1	1,2	8,5,2	1,2,3	0.7911	0.8000
18	(0,1,3,6)(0,4,9,16)(0,8,17,3)	10,7	1,2	4,8,4,1	1,2,3,4	0.7876	0.7941
19	(0,1,3,6)(0,4,9,16)(0,8,17,7)	12,6	1,2	6,6,6	1,2,3	0.7847	0.7917
22	(0,1,3,6)(0,5,12,8)(0,9,19,8)	18,3	1,2	10,9,2	1,2,3	0.7766	0.7857
23	(0,1,3,6)(0,5,12,8)(0,9,19,7)	20,2	1,2	10,10,2	1,2,3	0.7788	0.7841
26	(0,1,3,6)(0,4,9,16)(0,8,17,2)(0,4,16,3)	18,7	1,2	6,15,4	1,2,3	0.7764	0.7800
27	(0,1,3,6)(0,4,9,16)(0,8,17,2)(0,1,11,24)	20,6	1,2	6,18,2	1,2,3	0.7764	0.7788
30	(0,1,3,6)(0,4,9,16)(0,8,28,9)(0,12,25,10)	26,3	1,2	14,11,4	1,2,3	0.7707	0.7759
31	(0,1,3,6)(0,4,9,16)(0,8,29,9)(0,12,25,8)*	28,2	1,2	12,18	1,2	0.7726	0.775
34	(0,1,3,6)(0,4,9,16)(0,8,32,9)(0,12,25,5)(0,4,19,2)	26,7	1,2	14,13,4,2	1,2,3,4	0.7669	0.7727
35	(0,1,3,6)(0,4,9,16)(0,8,33,9)(0,12,25,4)(0,6,21,3)	28,6	1,2	12,20,2	1,2,4	0.7680	0.7721
38	(0,1,3,6)(0,4,9,16)(0,8,36,9)(0,26,1,15)(0,17,35,16)	34,3	1,2	20,13,2,2	1,2,3,4	0.7645	0.7703
39	(0,1,3,6)(0,4,9,16)(0,8,37,9)(0,27,1,15)(0,17,35,15)	36,2	1,2	22,10,6	1,2,3	0.7657	0.7697
42	(0,1,3,6)(0,4,9,16)(0,8,40,9)(0,30,1,15)(0,17,35,12) (0,2,22,1)	34,7	1,2	22,11,6,2	1,2,3,5	0.7617	0.7683
43	(0,1,3,6)(0,4,9,16)(0,8,41,9)(0,31,1,15)(0,17,35,11) (0,3,23,1)	36,6	1,2	20,16,4,2	1,2,3,4	0.7631	0.7679
46	(0,1,3,6)(0,4,9,16)(0,8,44,9)(0,34,1,15)(0,29,1,20) (0,23,45,20)	42,3	1,2	28,11,4,2	1,2,3,5	0.7603	0.7667
47	(0,1,3,6)(0,4,9,16)(0,8,45,9)(0,35,1,15) (0,30,1,20)(0,23,45,19)	44,2	1,2	26,16,2,2	1,2,3,4	0.7617	0.7663
50	(0,1,3,6)(0,4,9,16)(0,8,48,9)(0,38,1,15) (0,33,1,20)(0,23,45,16)(0,24,49,22)	42,7	1,2	26,15,6,2	1,2,3,5	0.7599	0.7653

*PBIBD (2), **BIBD

Table 5: Examination of GN_2-designs by Kedia and Misra [4].

V	Initial Block(s)	A-value (Sum of v-1 Eigen values	D-value (product of v-1 Eigen values)	GN_2- designs		PBIBD designs		E	UB
				$n(s)$	$\lambda(s)$	$n(s)$	$\lambda(s)$		
4	[4]: Theorem 2.1 t =1 IB = (0,3,1,2)**	8	16	2,1	2,4	3	4	1.0000	1.0000
6	[4]: Theorem 2.2 t 1 IB = (0,4,2,3)*	12	72	2,3	1,2	3,2	2,3	0.8929	0.9000
6	[4]: Theorem 2.3 t =1 IB = (1,0,2,4)*	12	72	2,3	1,2	3,2	2,3	0.8929	0.9000
7	[4]: Theorem 2.4 t =1 IB = (0,5,3,1)	14	49	2,2,2	0,1,2	2,2,2	1,2,3	0.8515	0.8750
	Little Adjustment in Above IB = (0,1,5,3)*			4,2	1,2	2,2,2	1,2,3	0.8515	0.8750
8	[8]: Example 5.2.1.1 IB = (0,7,1,4)*	16	256	6,1	1,2	2,5	1,2	0.8498	0.8571

*PBIBD (2), **BIBD

Table 6: Comparison between Existing and New GN_2-Designs for k = 4.

V	Reference	Association scheme	Efficiency	Upper bound	% of Upper bound
12	[8], Series 2.2, t = 2, w = 0	4	0.7992	0.8182	97.68
	Zafaryabet al. (2010), Theorem 3.4, t = 1	3	0.8054		98.43
	New	2	0.8159		99.72
14	[8], Series 2.3, t = 2, w = 0	3	0.7997	0.8077	99.01
	Zafaryabet al. (2010), Theorem 3.6, t = 1	3	0.7886		97.63
	[6], Table 2.3	3	0.7931		98.19
	New	2	0.8054		99.71
15	[8], Series 2.3, t = 2, w = 1	3	0.7952	0.8036	98.95
	[7], Theorem 3.7, t = 1	3	0.7947		98.89
	[6], Table 2.3	3	0.7947		98.89
	New	2	0.8003		99.56
20	[8], Series 2.2, t = 3, w = 0	3	0.7745	0.7895	98.10
	[7], Theorem 3.4, t = 2	3	0.7745		98.10
	New	2	0.7884		99.86
21	[8], Series 2.2, t = 2, w = 1	3	0.7716	0.7875	97.98
	[7], Theorem 3.5, t = 2	4	0.7766		98.61
	New	2	0.7854		99.73
22	[8], Series 2.3, t = 3, w = 0	3	0.7759	0.7857	98.75
	[7], Theorem 3.6, t = 2	3	0.7688		97.85
	[6], Table 2.3	3	0.7766		98.84
	New	2	0.7832		99.68
23	[8], Series 2.3, t = 3, w = 1	4	0.7723	0.7841	98.49
	[7], Theorem 3.7, t = 2	3	0.7738		98.68
	[6], Table 2.3	3	0.7788		99.32
	New	2	0.7810		99.60
24	[7], Theorem 3.8, t = 2	3	0.7723	0.7826	98.68
	New	2	0.7810		99.80
26	[8], Series 2.1, t = 3, w = 0	5	0.7678	0.7800	98.43
	[7], Theorem 3.2, t = 3	4	0.7726		99.05
	[6], Table 2.3	3	0.7764		99.54
	New	2	0.7793		99.91
27	[8], Series 2.1, t = 3, w = 1	3	0.7697	0.7788	98.83
	[7], Theorem 3.3, t = 3	4	0.7678		98.59
	[6], Table 2.3	3	0.7764		99.69
	New	2	0.7774		99.82
28	[8], Series 2.2, t = 4, w = 0	4	0.7613	0.7778	97.88
	[7], Theorem 3.4, t = 3	3	0.7622		97.99
	New	2	0.7764		99.82
29	[8], Series 2.2, t = 4, w = 1	3	0.7613	0.7768	98.00
	[7], Theorem 3.5, t = 3	3	0.7669		98.72
	New	2	0.7754		99.82
30	[8], Series 2.3, t = 4, w = 0	3	0.7650	0.7759	98.59
	[7], Theorem 3.7, t = 3	3	0.7603		97.99
	[6], Table 2.3	3	0.7707		99.33
	New	2	0.7735		99.69
32	[7], Theorem 3.8, t = 3	3	0.7641	0.7742	98.69
	New	2	0.7716		99.66
34	[8], Series 2.1, t = 4, w = 0	4	0.7610	0.7727	98.48
	[7], Theorem 3.2, t = 4	4	0.7645		98.94
	[6], Table 2.3	4	0.7669		99.25
	New	2	0.7716		99.86
35	[8], Series 2.1, t = 4, w = 1	4	0.7610	0.7721	98.56
	[7], Theorem 3.3, t = 4	3	0.7622		98.72
	[6], Table 2.3	3	0.7680		99.47
	New	2	0.7704		99.80
36	[8], Series 2.2, t = 5, w = 0	3	0.7564	0.7714	98.05
	[7], Theorem 3.4, t = 4	3	0.7564		98.05

	New	2	0.7704		99.87
37	[8], Series 2.2, t = 5, w = 1	3	0.7564	0.7708	98.13
	[7], Theorem 3.5, t = 4	4	0.7599		98.58
	New	2	0.7692		99.79
38	[8], Series 2.3, t = 5, w = 0	3	0.7599	0.7703	98.65
	[7], Theorem 3.6, t = 4	3	0.7564		98.19
	[6], Table 2.3	4	0.7645		99.25
	New	2	0.7680		99.70
39	[8], Series 2.3, t = 5, w = 1	4	0.7587	0.7697	98.57
	[7], Theorem 3.7, t = 4	3	0.7587		98.57
	[6], Table 2.3	3	0.7657		99.48
	New	2	0.7680		99.78
40	[7], Theorem 3.8, t = 4	3	0.7587	0.7692	98.63
	New	2	0.7669		99.70
42	[8], Series 2.1, t = 5, w = 0	5	0.7562	0.7683	98.42
	[7], Theorem 3.2, t = 5	5	0.7590		98.79
	[6], Table 2.3	4	0.7617		99.14
	New	2	0.7673		99.87
43	[8], Series 2.1, t = 1, w = 1	3	0.7576	0.7679	98.66
	[7], Theorem 3.3, t = 5	4	0.7562		98.48
	[6], Table 2.3	4	0.7631		99.37
	New	2	0.7659		99.74
44	[8], Series 2.2, t = 6, w = 0	4	0.7521	0.7674	98.01
	[7], T Theorem 3.4, t = 5	3	0.7535		98.19
	New	2	0.7659		99.80
45	[8], Series 2.2, t = 6, w = 1	2	0.7535	0.7670	98.24
	[7], Theorem 3.5, t = 5	2	0.7562		98.59
	New	2	0.7659		99.85
46	[8], Series 2.3, t = 6, w = 0	3	0.7562	0.7667	98.63
	[7], Theorem 3.6, t = 5	3	0.7535		98.28
	[6], Table 2.3	4	0.7603		99.16
	New	2	0.7645		99.71
47	[8], Series 2.3, t = 6, w = 1	3	0.7562	0.7663	98.68
	[7], Theorem 3.7, t = 5	3	0.7562		98.68
	[6], Table 2.3	4	0.7617		99.40
	New	2	0.7645		99.76
48	[6], Table 2.3	3	0.7562	0.7660	98.72
	[7], Theorem 3.8, t = 5	3	0.7562		98.72
	New	2	0.7645		99.80
50	[8], Series 2.1, t = 6, w = 0	4	0.7535	0.7653	98.46
	[7], Theorem 3.2, t = 6	4	0.7567		98.87
	[6], Table 2.3	4	0.7599		99.29
	New	2	0.7648		99.93