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Pythagorean Fuzzy Weighted Averaging Aggregation Operator and its Application to Decision Making Theory

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ABSTRACT

The objective of the present work is two folded. Firstly, Pythagorean fuzzy weighted averaging aggregation operator has been introduced along with their several properties, namely idem- potency, boundedness and monotonicity. Secondly, we applied this proposed operator to deal with multiple attribute group decision making problems under Pythagorean fuzzy information. For this we constructed an algorithm for multiple attribute group decision making problems. Finally, we constructed a numerical example for multiple attribute group decision making.

1. Introduction

The idea of fuzzy set was familiarized by Zadeh in 1965 [1]. In 1986, Atanassov presented the idea of intuitionistic fuzzy set (IFS), which is a general form of the fuzzy set [2]. The intuitionistic fuzzy set has gained increasing importance since its development [3-11]. Bustince and Burillo [12] demonstrated that vague sets are mathematically equal to intuitionistic fuzzy set. Bustince et al. [13] hosted the notion of intuitionistic fuzzy generators and also deliberated corresponding of IFS from intuitionistic fuzzy generators. Yager [14] introduced the notion of Pythagorean fuzzy set. Chen and Tan [15], handled multi criteria fuzzy decision-making based on vague set. Xu [16] established several operators such as, intuitionistic fuzzy weighted averaging (IFWA), intuitionistic fuzzy ordered weighted averaging (IFOWA) and intuitionistic fuzzy hybrid averaging (IFHA) operators. After the introduction of arithmetic aggregation operator, Xu and Yager [17] industrialized geometric aggregation operators, such as intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and intuitionistic fuzzy hybrid geometric (IFHG) operators. They also applied them to multiple attribute group decision making (MAGDM) based on intuitionistic fuzzy set (IFS). Wei [18] introduced the notion of the induced geometric aggregation operators with intuitionistic fuzzy ideal (IFI) and they also used these operators for group decision making. Wang and Liu [19] introduced the notion of intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operators. Zhang and Xu [20] developed Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method for multiple attribute group decision making. Intuitionistic fuzzy set have received much attention [21-23]. Yager [24] introduced the notion of Pythagorean fuzzy set. Xu and Yager [25-27] also worked in the field of intuitionistic aggregation operators. Rahman et al. [28-30] introduced some aggregation operators and also applied them to group decision making using Pythagorean fuzzy information.

Thus keeping the advantage of the above aggregation operators in this work we familiarize the notion of Pythagorean fuzzy weighted averaging aggregation operator and also discuss some of their basic properties.

This paper consists of six sections. In section 2, we give some core explanations and effects which can be used in our discussions later. In section 3, we develop Pythagorean fuzzy weighted averaging (PFWA) operator and also develop some of their properties. Section 4, contains an algorithm for multiple attribute group decision making (MAGDM). In section 5, we have conclusions.

2. Preliminaries

Definition 2.1 [1] Let I be a universal set and then fuzzy set can be defined as:

$$B=\left\{\left(i,\mu_B(i)\right)\mid i\in I\right\} \tag{1}$$

where μ_B is mapping from I to [0, 1] and $\mu_B(i)$ is said to be the degree of membership of element i in I.

Definition 2.2 [2] Let Z be a fixed set, then an intuitionistic fuzzy set can be defined as:

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$$L = \{(z, \mu_L(z), \eta_L(z)) \mid z \in Z\} \quad (2)$$

where $\mu_L(z)$ and $\eta_L(z)$ are mappings from Z to [0, 1] with some conditions such as:

$$0 \le \mu_L(z) + \eta_L(z) \le 1, \forall z \in Z$$

Definition 2.3 [14] Let K be a universal set, then a Pythagorean fuzzy set, can be defined as:

$$P = \{\langle k, u_P(k), v_P(k)\rangle \mid k \in K\} \quad (3)$$

where $u_P(k) : P \to [0,1], v_P(k) : K \to [0,1]$ are called membership and non-membership functions of $k \in K$ respectively, with condition $0 \le (u_P(k))^2 + (v_P(k))^2 \le 1$, for all $k \in K$. Let $\pi_P(k) = \sqrt{1 - u_P^2(k) - v_P^2(k)}$, then it is named the Pythagorean fuzzy index of $k \in K$ with condition $0 \le \pi_P(k) \le 1$, for every $k \in K$

Definition 2.4 [20] Let $\rho = (\mu_\rho, \eta_\rho)$, $\rho_1 = (\mu_{\rho_1}, \eta_{\rho_1})$, $\rho_2 = (\mu_{\rho_2}, \eta_{\rho_2})$, are three PFNs *and* $\Upsilon > 0$, *then*

(1) $\rho^c = (\eta_\rho, \mu_\rho)$,

(2) $\rho_1 \oplus \rho_2 = \left(\sqrt{\mu_{\rho_1}^2 + \mu_{\rho_2}^2 - \mu_{\rho_1}^2 \mu_{\rho_2}^2}, \eta_{\rho_1}\eta_{\rho_2}\right)$,

(3) $\rho_1 \otimes \rho_2 = \left(\mu_{\rho_1}\mu_{\rho_2}, \sqrt{\eta_{\rho_1}^2 + \eta_{\rho_2}^2 - \eta_{\rho_1}^2 \eta_{\rho_2}^2}\right)$,

(4) $\Upsilon\rho = \left(\sqrt{1 - \left(1 - \mu_\rho^2\right)^\Upsilon}, \eta_\rho^\Upsilon\right)$,

(5) $\rho^\Upsilon = \left(\mu_\rho^\Upsilon, \sqrt{1 - \left(1 - \eta_\rho^2\right)^\Upsilon}\right)$.

Definition 2.5 [20] Let $\rho = (\mu_\rho, \eta_\rho)$ be a Pythagorean fuzzy value, then we can find the score of ρ as following:

$$S(\rho) = \mu_\rho^2 - \eta_\rho^2 \quad (4)$$

where $S(\rho) \in [-1,1]$.

Definition 2.6 [20] Let $\rho = (\mu_\rho, \eta_\rho)$ be a PFN then the accuracy degree ρ can be defined as follows:

$$H(\rho) = \mu_\rho^2 + \eta_\rho^2 \quad (5)$$

where $H(\rho) \in [0,1]$.

Definition 2.7 [20] Let $\rho_1 = (\mu_{\rho_1}, \eta_{\rho_1})$ and $\rho_2 = (\mu_{\rho_2}, \eta_{\rho_2})$ be the two Pythagorean fuzzy numbers, then $S(\rho_1) = \mu_{\rho_1}^2 - \eta_{\rho_1}^2$, $S(\rho_2) = \mu_{\rho_2}^2 - \eta_{\rho_2}^2$, $H(\rho_1) = \mu_{\rho_1}^2 + \eta_{\rho_1}^2$, $H(\rho_2) = \mu_{\rho_2}^2 + \eta_{\rho_2}^2$ are the scores and accuracy of ρ_1 and ρ_2 respectively, then the following holds:

1. If $S(\rho_2) \succ S(\rho_1)$, then ρ_2 is greater than ρ_1, represented by $\rho_1 < \rho_2$,
2. If $S(\rho_1) = S(\rho_2)$, then

a. If $H(\rho_1) = H(\rho_2)$, then, ρ_1 and ρ_2 have the same information i.e., $\mu_{\rho_1} = \mu_{\rho_2}$ and $\eta_{\rho_1} = \eta_{\rho_2}$ represented by $\rho_1 = \rho_2$.

b. If $H(\rho_1) < H(\rho_2)$ then ρ_2 is greater than ρ_1

3. Pythagorean Fuzzy Weighted Averaging Aggregation Operator

Pythagorean fuzzy weighted averaging aggregation operator was introduced in Ref. [14] but in this paper we familiarize with their properties.

Definition 3.1: [14] Let $\rho_j = (\mu_{\rho_j}, \eta_{\rho_j})$ $(j = 1, 2, ..., n)$ be PFVs, and let $PFWA : \Omega^n \to \Omega$, then the Pythagorean fuzzy weighted averaging aggregation operator can be defined as:

$$PFWA_\omega(\rho_1, \rho_2, ..., \rho_n) = \omega_1\rho_1 \oplus \omega_2\rho_2 \oplus ... \oplus \omega_n\rho_n \quad (6)$$

Where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weighted vector of ρ_j with condition $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. If $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the PFWA operator is converted to PFA operator which is defined as:

$$PFA(p_1, p_2, ..., p_n) = \frac{1}{n}(\rho_1 \oplus \rho_2 \oplus ... \oplus \rho_n) \quad (7)$$

Example 3.2: Let

$$\rho_1 = (0.7, 0.6), \rho_2 = (0.5, 0.7)$$
$$\rho_3 = (0.6, 0.4), \rho_4 = (0.8, 0.5)$$

and $w = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of ρ_j $(j = 1, 2, 3, 4)$, *then*

$$PFWA_w(\rho_1,\rho_2,\rho_3,\rho_4) = \left(\sqrt{1-\prod_{j=1}^{4}\left(1-\mu_{\rho_j}^2\right)^{w_j}}, \prod_{j=1}^{4}\left(\eta_{\rho_j}\right)^{w_j}\right) = (0.6978, 0.5093).$$

Theorem 3.3: Let $\rho_j = \left(\mu_{\rho_j}, \eta_{\rho_j}\right)$ $(j = 1,2,...,n)$ are PFVs, then their aggregated value by applying PFWA operator is also a PFV

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_n) = \left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{n}\eta_{\rho_j}^{\omega_j}\right) \quad (8)$$

And also the weighted vector of ρ_j $(j = 1,2,...,n)$ is $\omega = (\omega_1,\omega_2,...,\omega_n)^T$ with some conditions $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Proof: By mathematical induction we can prove that equation (8) holds for all *n*. First we show that equation (8) holds for *n*=2, since

$$\omega_1\rho_1 = \left(\sqrt{1-\left(1-\mu_{\rho_1}^2\right)^{\omega_1}}, \eta_{\rho_1}^{\omega_1}\right)$$

$$\omega_2\rho_2 = \left(\sqrt{1-\left(1-\mu_{\rho_2}^2\right)^{\omega_2}}, \eta_{\rho_2}^{\omega_2}\right)$$

So

$$\omega_1\rho_1 \oplus \omega_2\rho_2 = \left(\sqrt{1-\left(1-\mu_{\rho_1}^2\right)^{\omega_1}}, \eta_{\rho_1}^{\omega_1}\right) \oplus \left(\sqrt{1-\left(1-\mu_{\rho_1}^2\right)^{\omega_1}}, \eta_{\rho_1}^{\omega_1}\right)$$

$$= \left(\sqrt{\begin{array}{c}1-\left(1-\mu_{\rho_1}^2\right)^{\omega_1}+1-\left(1-\mu_{\rho_2}^2\right)^{\omega_2}\\ -\left(1-\left(1-\mu_{\rho_1}^2\right)^{\omega_1}\right)\left(1-\left(1-\mu_{\rho_2}^2\right)^{\omega_2}\right)\end{array}}, \eta_{\rho_1}^{\omega_1}\eta_{\rho_2}^{\omega_2}\right)$$

$$= \left(\sqrt{1-\prod_{j=1}^{2}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{2}\eta_{\rho_j}^{\omega_j}\right)$$

Thus equation (8) is true for *n*=2, let us suppose that Eq. (8) is true for *n*=*k*, then we have

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_k) = \left(\sqrt{1-\prod_{j=1}^{k}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{k}\eta_{\rho_j}^{\omega_j}\right)$$

Now we show that equation (8) is true for *n=k+1*.

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_{k+1}) = \left(\sqrt{1-\prod_{j=1}^{k}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{k}\eta_{\rho_j}^{\omega_j}\right) \oplus \left(\sqrt{1-\left(1-\mu_{\rho_{k+1}}^2\right)^{\omega_{k+1}}}, \left(\eta_{\rho_{k+1}}\right)^{\omega_{k+1}}\right)$$

$$= \left(\sqrt{1-\prod_{j=1}^{k+1}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{k+1}\eta_{\rho_j}^{\omega_j},\right)$$

Hence equation (8) holds for *n=k+1*. Thus Eq. (8) holds for all *n*.

Theorem 3.4: Let $\rho_j = \left(\mu_{\rho_j}, \eta_{\rho_j}\right)$ $(j = 1,2,3,...,n)$ be the PFVs and the weighted vector of ρ_j $(j = 1,2,...,n)$ is $\omega = (\omega_1,\omega_2,...,\omega_n)^T$ with some conditions $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$. If ρ_j $(j = 1,2,...,n)$ are mathematically equal, then:

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_n) = \rho. \quad (9)$$

Proof: As we know that:

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_n) = \omega_1\rho_1 \oplus \omega_2\rho_2 \oplus ... \oplus \omega_n\rho_n.$$

Let ρ_j $(j = 1,2,3,...,n) = \rho$, then:

$$PFWA_\omega(\rho_1,\rho_2,...,\rho_n) = \omega_1\rho_1 \oplus \omega_2\rho_2 \oplus ... \oplus \omega_n\rho_n = (\rho)^{\sum_{j=1}^{n}\omega_j} = \rho.$$

Theorem 3.5: Let $\rho_j = \left(\mu_{\rho_j}, \eta_{\rho_j}\right)$ $(j = 1,2,...,n)$ be PFVs and let the weighted vector of ρ_j be $\omega = (\omega_1,\omega_2,...,\omega_n)^T$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

If,

$$\rho^- = \left(\min_j \left(\mu_{\rho_j}\right), \max_j \left(\eta_{\rho_j}\right) \right),$$

$$\rho^+ = \left(\max_j \left(\mu_{\rho_j}\right), \min_j \left(\eta_{\rho_j}\right) \right).$$

Then:

$$\rho^- \le PFWA_\omega\left(\rho_1, \rho_2, ..., \rho_n\right) \le \rho^+ \qquad (10)$$

Proof: As we know that:

$$\min_j \left(\mu_{\rho_j}\right) \le \mu_{\rho_j} \le \max_j \left(\mu_{\rho_j}\right) \qquad (11)$$

$$\min_j \left(\eta_{\rho_j}\right) \le \eta_{\rho_j} \le \max_j \left(\eta_{\rho_j}\right) \qquad (12)$$

From equation (11) we have:

$$\Leftrightarrow \sqrt{\min_j \left(\mu_{\rho_j}\right)^2} \le \sqrt{\left(\mu_{\rho_j}\right)^2} \le \sqrt{\max_j \left(\mu_{\rho_j}\right)^2}$$

$$\Leftrightarrow \sqrt{\left(1 - \max_j \left(\mu_{\rho_j}\right)^2\right)^{w_j}} \le \sqrt{\left(1 - \mu_{\rho_j}^2\right)^{w_j}} \le \sqrt{\left(1 - \min_j \left(\mu_{\rho_j}\right)^2\right)^{w_j}}$$

$$\Leftrightarrow \sqrt{\left(-1 + \min_j \left(\mu_{\rho_j}\right)^2\right)} \le \sqrt{-\prod_{j=1}^{n} \left(1 - \mu_{\rho_j}^2\right)^{w_j}} \le \sqrt{\left(-1 + \max_j \left(\mu_{\rho_j}\right)^2\right)}$$

$$\Leftrightarrow \min_j \left(\mu_{\rho_j}\right) \le \sqrt{1 - \prod_{j=1}^{n} \left(1 - \mu_{\rho_j}^2\right)^{w_j}} \le \max_j \left(\mu_{\rho_j}\right). \qquad (13)$$

Now from equation (12) we have

$$\Leftrightarrow \min_j \left(\eta_{\bar{a}_j}\right)^{w_j} \le \left(\eta_{\bar{a}_j}\right)^{w_j} \le \max_j \left(\eta_{\bar{a}_j}\right)^{w_j}$$

$$\Leftrightarrow \prod_{j=1}^{n} \min_j \left(\eta_{\rho_j}\right)^{w_j} \le \prod_{j=1}^{n} \left(\eta_{\rho_j}\right)^{w_j} \le \prod_{j=1}^{n} \max_j \left(\eta_{\rho_j}\right)^{w_j}$$

$$\Leftrightarrow \min_j \left(\eta_{\rho_j}\right)^{\sum_{j=1}^{n} w_j} \le \prod_{j=1}^{n} \left(\eta_{\rho_j}\right)^{w_j} \le \max_j \left(\eta_{\rho_j}\right)^{\sum_{j=1}^{n} w_j}$$

$$\Leftrightarrow \min_j \left(\eta_{\rho_j}\right) \le \prod_{j=1}^{n} \left(\eta_{\rho_j}\right)^{w_j} \le \max_j \left(\eta_{\rho_j}\right). \qquad (14)$$

Let $PFWA_w\left(\rho_1, \rho_2, ..., \rho_n\right) = \rho = \left(\mu_\rho, \eta_\rho\right)$, then,

$$S(\rho) = \mu_\rho^2 - \eta_\rho^2 \le \max_j \left(\mu_\rho\right)^2 - \min_j \left(\eta_\rho\right)^2 = S\left(\rho^+\right)$$

Thus $S(\rho) \le S\left(\rho^+\right)$. Again,

$$S(\rho) = \mu_\rho^2 - \eta_\rho^2 \ge \min_j \left(\mu_\rho\right)^2 - \max_j \left(\eta_\rho\right)^2 = S\left(\rho^-\right).$$

Thus $S(\rho) \ge S\left(\rho^-\right)$. If $S(\rho) < S\left(\rho^+\right)$ and $S(\rho) > S\left(\rho^-\right)$. Then,

$$\rho^- < PFWG_w\left(\rho_1, \rho_2, ..., \rho_n\right) < \rho^+. \qquad (15)$$

If $S(\rho) = S\left(\rho^+\right)$,

then:

$$\Leftrightarrow \mu_\rho^2 - \eta_\rho^2 = \max_j \left(\mu_{\rho_j}\right)^2 - \min_j \left(\eta_{\rho_j}\right)^2$$

$$\Leftrightarrow \mu_\rho^2 = \max_j \left(\mu_{\rho_j}\right)^2, \eta_\rho^2 = \min_j \left(\eta_{\rho_j}\right)^2$$

$$\Leftrightarrow \mu_\rho = \max_j \left(\mu_{\rho_j}\right), \eta_\rho = \min_j \left(\eta_{\rho_j}\right).$$

Since:

$$H(\rho) = \mu_\rho^2 + \eta_\rho^2 = \max_j \left(\mu_{\rho_j}\right)^2 + \min_j \left(\eta_{\rho_j}\right)^2 = H\left(\rho^+\right).$$

Thus:

$$PFWA_w\left(\rho_1, \rho_2, ..., \rho_n\right) = \rho^+. \qquad (16)$$

If $S(\rho) = S\left(\rho^-\right)$, then:

$$\Leftrightarrow \mu_\rho^2 - \eta_\rho^2 = \min_j \left(\eta_{\rho_j}\right)^2 - \max_j \left(\mu_{\rho_j}\right)^2$$

$$\Leftrightarrow \mu_\rho^2 = \min_j \left(\eta_{\rho_j}\right)^2, \eta_\rho^2 = \max_j \left(\mu_{\rho_j}\right)^2$$

$$\Leftrightarrow \mu_\rho = \min_j \left(\eta_{\rho_j}\right), \eta_\rho = \max_j \left(\mu_{\rho_j}\right).$$

Since

$$H(\rho) = \mu_\rho^2 + \eta_\rho^2 = \min_j \left(\eta_{\rho_j}\right)^2 + \max_j \left(\mu_{\rho_j}\right)^2 = H\left(\rho^-\right).$$

Thus

$$PFWA_\omega\left(\rho_1, \rho_2, ..., \rho_n\right) = \rho^-. \qquad (17)$$

Thus from equation (15) to (17), we have:

$$\rho^- \le PFWA_\omega\left(\rho_1, \rho_2, ..., \rho_n\right) \le \rho^+.$$

Theorem 3.6 Let $\rho_j = \left(\mu_{\rho_j}, \eta_{\rho_j}\right)(j = 1, 2, 3, ..., n)$ and $\rho_j^* = \left(\mu_{\rho_j^*}, \eta_{\rho_j^*}\right)(j = 1, 2, 3, ..., n)$ be the two collection of PFVs . If $\mu_{\rho_j} \le \mu_{\rho_j^*}$ and $\eta_{\rho_j} \ge \eta_{\rho_j^*}$.

Then

$$PFWA_{\omega}(\rho_1,\rho_2,...,\rho_n) \le PFWA_{\omega}\left(\rho_1^*,\rho_2^*,...,\rho_n^*\right)$$

Proof: Since, $\mu_{\rho_j} \le \mu_{\rho_j^*}$ and $\eta_{\rho_j} \ge \eta_{\rho_j^*}$. Then

$$\Leftrightarrow \mu_{\rho_j}^2 \le \mu_{\rho_j^*}^2$$

$$\Leftrightarrow \sqrt{1-\mu_{\rho_j^*}^2} \le \sqrt{1-\mu_{\rho_j}^2}$$

$$\Leftrightarrow \sqrt{\left(1-\mu_{\rho_j^*}^2\right)^{w_j}} \le \sqrt{\left(1-\mu_{\rho_j}^2\right)^{w_j}}$$

$$\Leftrightarrow \sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\rho_j}^2\right)^{w_j}} \le \sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\rho_j^*}^2\right)^{w_j}}. \quad (19)$$

Now $\eta_{\rho_j} \ge \eta_{\rho_j^*}$.

$$\eta_{\rho_j}^{w_j} \ge \eta_{\rho_j^*}^{w_j} \Leftrightarrow \prod_{j=1}^{n}\eta_{\rho_j}^{w_j} \ge \prod_{j=1}^{n}\eta_{\rho_j^*}^{w_j}. \quad (20)$$

Let

$$PFWA_w(\rho_1,\rho_2,\rho_3,...,\rho_n) = \rho.$$

and

$$PFWA_w\left(\rho_1^*,\rho_2^*,\rho_3^*,...,\rho_n^*\right) = \rho^*$$

Then from equations (19) and (20), we have:

$$S(\rho) \le S\left(\rho^*\right).$$

If,

$$S(\rho) < S\left(\rho^*\right).$$

Then

$$PFWA_w(\rho_1,\rho_2,...,\rho_n) < PFWA_w\left(\rho_1^*,\rho_2^*,...,\rho_n^*\right). \quad (21)$$

If

$$S(\rho) = S\left(\rho^*\right).$$

Then

$$\Leftrightarrow \mu_{\rho_j}^2 - \eta_{\rho_j}^2 = \mu_{\rho_j^*}^2 - \eta_{\rho_j^*}^2$$

$$\Leftrightarrow \mu_{\rho_j}^2 = \mu_{\rho_j^*}^2, \eta_{\rho_j}^2 = \eta_{\rho_j^*}^2$$

$$\Leftrightarrow \mu_{\rho_j} = \mu_{\rho_j^*}, \eta_{\rho_j} = \eta_{\rho_j^*}.$$

Since

$$H(\rho) = \mu_{\rho_j}^2 + \eta_{\rho_j}^2 = \mu_{\rho_j^*}^2 + \eta_{\rho_j^*}^2 = H\left(\rho^*\right).$$

Thus

$$PFWA_w(\rho_1,\rho_2,...,\rho_n) = PFWA_w\left(\rho_1^*,\rho_2^*,...,\rho_n^*\right). \quad (22)$$

Thus from equation (21) and (22) we have

$$PFWA_w(\rho_1,\rho_2,...,\rho_n) \le PFWA_w\left(\rho_1^*,\rho_2^*,...,\rho_n^*\right).$$

Example: 3.7 Let

$$\rho_1 = (0.5,0.7), \rho_2 = (0.3,0.8),$$
$$\rho_3 = (0.5,0.6), \rho_4 = (0.5,0.5)$$

and

$$\rho^*{}_1 = (0.6,0.4), \rho*_2 = (0.7,0.5),$$
$$\rho^*{}_3 = (0.8,0.3), \rho^*{}_4 = (0.9,0.2),$$

where $\omega = (0.1,0.2,0.3,0.4)$.

Now using the PFWA operator we get the following result.

$$PFWA_w(\rho_1,\rho_2,\rho_3,\rho_4)$$
$$= \left(\sqrt{1-\prod_{j=1}^{4}\left(1-\mu_{\rho_j}^2\right)^{\omega_j}}, \prod_{j=1}^{4}\eta_{\rho_j}^{\omega_j}\right)$$
$$= (0.469, 0.600).$$

Again

$$PFWA_w\left(\rho^*{}_1,\rho^*{}_2,\rho^*{}_3,\rho^*{}_4\right)$$
$$= \left(\sqrt{1-\prod_{j=1}^{4}\left(1-\mu_{\rho^*{}_j}^2\right)^{\omega_j}}, \prod_{j=1}^{4}\eta_{\rho^*{}_j}^{\omega_j}\right)$$
$$= (0.8267, 0.2907).$$

4. Application of the Pythagorean fuzzy Weighted Averaging Aggregation Operator to Multiple Attribute Group Decision Making

Algorithm Let $H = \{H_1,H_2,H_3,...,H_n\}$ be a set of n alternatives, and $G = \{G_1,G_2,G_3,...,G_m\}$ be the set of m attributes and $w = (w_1,w_2,w_3,...,w_m)^T$ be the weighted vector of the attributes $G_i\,(i = 1,2,3,...,m)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{m} w_i = 1$.

Step 1: In this step the decision makers provide the information in the form of a matrix.

Step 2: Compute $\rho_j\,(j = 1,2,3,...,n)$ using Pythagorean fuzzy weighted averaging PFWA aggregation operator.

Step 3: Compute the scores of $\rho_j\left(j=1,2,3,...,n\right)$. If there is no difference between two or more than two scores, then we must have to calculate the degrees of accuracy.

Step 4: Arrange the scores function of all alternatives in the form of descending order and select that alternative, which has the highest score function value.

Example 4.1 We consider an example for selecting a watch from different watches.

Suppose a customer wants to buy a watch from different watches, let $H_1, H_2, H_3, H_4, H_5,$ represent the five watches of different companies. Let $G_1, G_2, G_3,$ be the criteria of these watches. In the process of choosing one of the watches; three factors are considered, G_1: Price of each watch. G_2: Model of each watch. G_3: Design of each watch. Suppose the weight vector of $G_i\left(i=1,2,3\right)$ is $w=\left(0.2,0.3,0.5\right)^T$ and the Pythagorean fuzzy values of the alternative $H_j\left(j=1,2,3,4,5\right)$ are represented by the following decision matrix:

Step 1: The decision maker gives his decision in Table 1.

Table 1: Pythagorean fuzzy decision matrix

	H_1	H_2	H_3	H_4	H_5
G_1	(0.6, 0.5)	(0.7, 0.6)	(0.8, 0.4)	(0.8, 0.2)	(0.7, 0.5)
G_2	(0.5, 0.6)	(0.7, 0.5)	(0.7, 0.5)	(0.8, 0.5)	(0.7, 0.6)
G_3	(0.7, 0.3)	(0.7, 0.4)	(0.8, 0.2)	(0.9, 0.2)	(0.7, 0.3)

Step 2: Compute $\rho_j\left(j=1,2,3,4\right)$, by applying PFWA operator

$$\rho_1=\left(0.6330,0.4090\right),\rho_2=\left(0.7000,0.4637\right)$$
$$\rho_3=\left(0.7748,0.3023\right),\rho_4=\left(0.8593,0.2632\right)$$
$$\rho_5=\left(0.7000,0.4090\right)$$

Step 3: in this step we can find the scores of $\rho_j\left(j=1,2,3,4,5\right)$

$$S\left(\rho_1\right)=\left(0.6330\right)^2-\left(0.4090\right)^2=0.2334$$
$$S\left(\rho_2\right)=\left(0.7000\right)^2-\left(0.4637\right)^2=0.2750$$
$$S\left(\rho_3\right)=\left(0.7748\right)^2-\left(0.3023\right)^2=0.5089$$
$$S\left(\rho_4\right)=\left(0.8593\right)^2-\left(0.2632\right)^2=0.9661$$
$$S\left(\rho_5\right)=\left(0.7000\right)^2-\left(0.4090\right)^2=0.3227$$

Step 4: Arrange the scores of the all alternatives in the form of descending order and select that alternative, which has the highest score function. Since $\rho_4>\rho_3>\rho_5>\rho_2>\rho_1$. Hence $H_4>H_3>H_5>H_2>H_1$. Thus H_4 is the best option for the customer.

5. Conclusion

The objective of this paper is to present an aggregation operator based on Pythagorean fuzzy number and applied them to the multi-attribute decision making problem, where attribute values are Pythagorean fuzzy numbers. Firstly, we have developed Pythagorean fuzzy weighted averaging (PFWA) aggregation operator along with their properties namely, idem potency, boundedness and monotonicity**.** Finally, we have developed a method for multi-criteria decision making based on the proposed operator and the operational processes have illustrated in detail. The main advantage of using the proposed method and operator is that this method provides more general, accurate and precise results. Therefore, the suggested methodology can be used for any type of selection problem involving any number of selection attributes. Therefore, this method plays a vital role in real world problems. We ended the paper with an application of the new approach in a decision making problem.

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